Test 2 - Abstract Algebra

Dr. Graham-Squire, Spring 2016

Name: ______________________________

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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(*signature*)

DIRECTIONS

1. Don't panic.
2. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
3. Cell phones and computers are not allowed on this test. Calculators are allowed, though it is unlikely that they will be helfpul.
4. If you are confused about what a particular notation means (e.g. $U(n)$) or whether or not something can be assumed (as opposed to needing to prove it), feel free to ask. Note: S_n is the group of permutations of the numbers 1 through n, D_m is the group of symmetries of a regular m-sided figure, and $U(k)$ is the group of positive integers less than k which are relatively prime to k, under multiplication
5. You must to all of the first four questions, but only two of the last three (if you do all of the last three questions, I will grade them all and give you the two highest scores of the three).
6. Make sure you sign the pledge above.
7. Number of questions = 6. Total Points = 30.

1. (5 points) Some of the following six groups are isomorphic and others are not isomorphic. If a group is not isomorphic to other groups, give a (brief) explanation of why. If two groups *are* isomorphic, give a brief explanation of why (full proof is not necessary).

$$S_4 \qquad D_{12} \qquad \mathbb{Z}_3 \oplus \mathbb{Z}_8 \qquad \mathbb{Z}_{24} \qquad U(5) \oplus \mathbb{Z}_6 \qquad \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$$

2. (4 points) Give at least *two* reasons why $\mathbb{Z}$ (under addition) is not isomorphic to $\mathbb{R}$ (under multiplication).

3. (5 points) Let G be the following subgroup of S_6:

$$G = \{(1), (12)(34), (1234)(56), (13)(24), (1432)(56), (56)(13), (14)(23), (24)(56)\}.$$

Recall that the *stabilizer* is the set of elements of the permutation group that send a number to itself, and the *orbit* is all of the numbers that a particular number can get sent to.

(a) Find the stabilizer of 1 (in G) and the orbit of 1 (in G).

(b) Find the stabilizer of 5 (in G) and the orbit of 5 (in G).

(c) In what way(s) do your answers above confirm or refute the orbit-stabilizer theorem?

4. (4 points) Let $G = U(15) \oplus \mathbb{Z}_{30} \oplus S_8$. Find the order of the element

$$(2, 7, (123)(154)) \in G.$$

Explain your reasoning.

For the next three problems, you will receive the highest 2 scores out of the three, so you do NOT have to answer all of them.

5. (6 points) Let H be a subgroup of G. Prove that, for $a \in G$,

$$aH = H \text{ if and only if } a \in H.$$

6. (6 points) Prove the following for groups G and H:

$$\text{If } G \oplus H \text{ is cyclic, then } G \text{ and } H \text{ are both cyclic.}$$

7. (6 points) Suppose $\alpha : G \to H$ is an isomorphism from G to H, and $\beta : H \to K$ is an isomorphism from H to K. Prove that G is isomorphic to K. (Note: you may have to use certain conclusions we have proved previously in this course (and Math Thought). If you are unsure whether you can state something or need to prove it, ask Dr. G-S).

8. (2 points) Extra Credit: Recall that $Inn(G)$ denotes the group of *inner* automorphisms of G, that is, automorphisms of the form ϕ_g where $g \in G$ and $\phi_g : G \to G$ is defined by $\phi_g(x) = gxg^{-1}$. Prove that $|Inn(G)| = 1$ if and only if G is Abelian.